

## Limit theorem for continuous-time quantum walk on the line

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Concerning a *discrete-time* quantum walk  $X_t^{(d)}$  with a symmetric distribution on the line, whose evolution is described by the Hadamard transformation, it was proved by the author that the following weak limit theorem holds:  $X_t^{(d)}/t \rightarrow dx/\pi(1-x^2)\sqrt{1-2x^2}$  as  $t \rightarrow \infty$ . The present paper shows that a similar type of weak limit theorem is satisfied for a *continuous-time* quantum walk  $X_t^{(c)}$  on the line as follows:  $X_t^{(c)}/t \rightarrow dx/\pi\sqrt{1-x^2}$  as  $t \rightarrow \infty$ . These results for quantum walks form a striking contrast to the central limit theorem for symmetric discrete- and continuous-time classical random walks:  $Y_t/\sqrt{t} \rightarrow e^{-x^2/2}dx/\sqrt{2\pi}$  as  $t \rightarrow \infty$ . The work deals also with the issue of the relationship between discrete and continuous-time quantum walks. This topic, subject of a long debate in the previous literature, is treated within the formalism of matrix representation and the limit distributions are exhaustively compared in the two cases.

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### I. INTRODUCTION

Quantum walks have recently been introduced and investigated, with the hope that they may be useful in constructing new efficient quantum algorithms (for reviews of quantum walks, see [1–3]). There are two distinct types of the quantum walk: one is a discrete-time case [4–12], the other is a continuous-time case [13–19]. The quantum walk can be considered as a quantum analog of the classical random walk. However, there are some differences between them. For the discrete-time symmetric *classical* random walk  $Y_t^o$  starting from the origin, the central limit theorem shows that  $Y_t^o/\sqrt{t} \rightarrow e^{-x^2/2}dx/\sqrt{2\pi}$  as  $t \rightarrow \infty$ . The same limit theorem holds for the continuous-time classical symmetric random walk (related results can be found in Refs. [20,21]). The limit density function is the normal distribution and has a bell-shaped curve with one peak at the center. On the other hand, concerning a *discrete-time* quantum walk  $X_t^{(d)}$  with a symmetric distribution on the line, whose evolution is described by the Hadamard transformation, it was shown by the author [8,9] that the following weak limit theorem holds:  $X_t^{(d)}/t \rightarrow dx/\pi(1-x^2)\sqrt{1-2x^2}$  as  $t \rightarrow \infty$ . This paper presents that a similar type of weak limit theorem is proved for a *continuous-time* quantum walk  $X_t^{(c)}$  on the line as follows:  $X_t^{(c)}/t \rightarrow dx/\pi\sqrt{1-x^2}$  as  $t \rightarrow \infty$ . Both limit density functions for quantum walks have two peaks at the two end points of the support. The study of weak limits for discrete-time quantum walks is treated also in Ref. [6], with a simplified proof with respect to earlier derivations.

As a corollary, we have the following result. Let  $\sigma_{(d)}^c(t)$  ( $\sigma_{(c)}^c(t)$ ) be the standard deviation of the probability distribution for a discrete-time (continuous-time) classical random walk on the line at time  $t$ . Similarly,  $\sigma_{(d)}^q(t)$  ( $\sigma_{(c)}^q(t)$ ) denotes the standard deviation for a discrete-time (continuous-time) quantum walk. Then, the central limit theorem implies that  $\sigma_{(d)}^c(t)$ ,  $\sigma_{(c)}^c(t) \asymp \sqrt{t}$ , where  $f(t) \asymp g(t)$  in-

dicates that  $f(t)/g(t) \rightarrow c_*(\neq 0)$  as  $t \rightarrow \infty$ . In contrast, using our limit theorems, it is shown that  $\sigma_{(d)}^q(t)$ ,  $\sigma_{(c)}^q(t) \asymp t$  hold. That is, the quantum walks spread over the line faster than the classical walks in both the discrete- and continuous-time cases.

The remainder of this paper is organized as follows. Section II gives the definition of the walk and our results. In Sec. III, we prove Proposition 1. Section IV is devoted to a proof of Theorem 1. Conclusion and discussion are given in Sec. V.

### II. MODEL AND RESULTS

Let  $\mathbf{Z}$  be the set of integers. To define the continuous-time quantum walk on  $\mathbf{Z}$ , we introduce an  $\infty \times \infty$  adjacency matrix of  $\mathbf{Z}$  denoted by  $A$  as follows:

$$A = \begin{matrix} \dots & -3 & -2 & -1 & 0 & +1 & +2 & \dots \\ \vdots & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ -3 & \dots & 0 & 1 & 0 & 0 & 0 & \dots \\ -2 & \dots & 1 & 0 & 1 & 0 & 0 & \dots \\ -1 & \dots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 1 & 0 & 1 & \dots \\ +1 & \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ +2 & \dots & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \end{matrix} .$$

The amplitude wave function of the walk at time  $t$ ,  $|\Psi(t)\rangle$ , is defined by

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle,$$

where

$$U(t) = e^{itA/2}.$$

Note that  $U(t)$  is a unitary matrix. As an initial state, we take

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$$|\Psi(0)\rangle = T(\dots, 0, 0, 0, 1, 0, 0, 0, \dots),$$

where  $T$  indicates the transposed operator. Concerning the details of the definitions for the continuous-time case, see [13,14,22]. Let  $|\Psi(k,t)\rangle$  be an amplitude wave function at location  $k$  at time  $t$ . The probability that the particle is at location  $k$  at time  $t$ ,  $P(k,t)$ , is given by

$$P(k,t) = \langle \Psi(k,t) | \Psi(k,t) \rangle.$$

As for results on the walk on a circle, i.e.,  $\{0, 1, \dots, N-1\}$ , see [13,14,18].

We will obtain an explicit form of  $U(t)$  first. Our approach is a direct computation based on the  $\infty \times \infty$  matrix  $A$  without using its eigenvalues and eigenvectors in order to clarify a relation (stated below) between the continuous-time and discrete-time quantum walks, (another approach and the same explicit form can be found in Sec. III C of Ref. [15]). Let  $J_k(x)$  denote the Bessel function of the first kind of order  $k$ . As for the Bessel function, see Watson [23] and Chapter 4 in Andrews *et al.* [24].

*Proposition 1.* In our setting, we have

$$U(t) = \begin{matrix} & & \dots & -3 & -2 & -1 & 0 & +1 & +2 & \dots \\ \vdots & \ddots & & & & & & & & \dots \\ -3 & \dots & J_0(t) & iJ_1(t) & i^2J_2(t) & i^3J_3(t) & i^4J_4(t) & i^5J_5(t) & \dots \\ -2 & \dots & iJ_1(t) & J_0(t) & iJ_1(t) & i^2J_2(t) & i^3J_3(t) & i^4J_4(t) & \dots \\ -1 & \dots & i^2J_2(t) & iJ_1(t) & J_0(t) & iJ_1(t) & i^2J_2(t) & i^3J_3(t) & \dots \\ 0 & \dots & i^3J_3(t) & i^2J_2(t) & iJ_1(t) & J_0(t) & iJ_1(t) & i^2J_2(t) & \dots \\ +1 & \dots & i^4J_4(t) & i^3J_3(t) & i^2J_2(t) & iJ_1(t) & J_0(t) & iJ_1(t) & \dots \\ +2 & \dots & i^5J_5(t) & i^4J_4(t) & i^3J_3(t) & i^2J_2(t) & iJ_1(t) & J_0(t) & \dots \\ \vdots & \dots & & & & & & & \ddots \end{matrix}$$

That is, the  $(l,m)$  component of  $U(t)$  is given by  $i^{|l-m|}J_{|l-m|}(t)$ .

From Proposition 1, looking at a column of  $U(t)$ , we have immediately

*Corollary 1.* The amplitude wave function of our model is given by

$$|\Psi(t)\rangle = T(\dots, i^3J_3(t), i^2J_2(t), iJ_1(t), J_0(t), iJ_1(t), i^2J_2(t), i^3J_3(t), \dots),$$

that is,  $|\Psi(k,t)\rangle = i^{|k|}J_{|k|}(t)$  for any location  $k \in \mathbf{Z}$  and time  $t \geq 0$ .

Moreover, noting that  $J_{-k}(t) = (-1)^k J_k(t)$  [Eq. (4.5.4) in Ref. [24]], we have

*Corollary 2.* The probability distribution is

$$P(k,t) = J_{|k|}^2(t) = J_k^2(t),$$

for any location  $k \in \mathbf{Z}$  and time  $t \geq 0$ .

In fact, the following result [see Eq. (4.9.5) in Ref. [24]]:

$$J_0^2(t) + 2 \sum_{k=1}^{\infty} J_k^2(t) = 1$$

ensures that  $\sum_{k=-\infty}^{\infty} P(k,t) = 1$  for any  $t \geq 0$ . Remark that the distribution is symmetric for any time, i.e.,  $P(k,t) = P(-k,t)$ .

One of the interesting points of the above-mentioned results is as follows. Let  $\mathbf{C}$  be the set of complex numbers. Fix a positive integer  $r$ . We suppose that a unitary matrix  $U_r$  has the following form:

$$U_r = \begin{matrix} & & \dots & -3 & -2 & -1 & 0 & +1 & +2 & \dots \\ \vdots & \ddots & & & & & & & & \dots \\ -3 & \dots & w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & \dots \\ -2 & \dots & w_{-1} & w_0 & w_1 & w_2 & w_3 & w_4 & \dots \\ -1 & \dots & w_{-2} & w_{-1} & w_0 & w_1 & w_2 & w_3 & \dots \\ 0 & \dots & w_{-3} & w_{-2} & w_{-1} & w_0 & w_1 & w_2 & \dots \\ +1 & \dots & w_{-4} & w_{-3} & w_{-2} & w_{-1} & w_0 & w_1 & \dots \\ +2 & \dots & w_{-5} & w_{-4} & w_{-3} & w_{-2} & w_{-1} & w_0 & \dots \\ \vdots & \dots & & & & & & & \ddots \end{matrix}$$

with  $w_k \in \mathbf{C}$  ( $k \in \mathbf{Z}$ ) and  $w_s = 0$  for  $|s| > r$ . Then *No-Go Lemma* (see Ref. [10]) shows that there is the only nonzero  $w_*$  with  $|w_*| = 1$ ; that is, there exists no nontrivial, homogeneous finite-range model governed by the  $U_r$ . For example, when  $r=1$ , we have “ $|w_{-1}|=1, w_0=w_1=0$ ,” “ $|w_0|=1, w_{-1}=w_1=0$ ,” or “ $|w_1|=1, w_0=w_{-1}=0$ .” Thus the model has a trivial probability distribution. However, our homogeneous, but *infinite-range* model has a nontrivial probability distribution given by squared Bessel functions (Corollary 2).

An open problem on quantum walks is to clarify a relation between discrete-time and continuous-time quantum walks (see Ref. [1], for example). Another interesting point of the results is to shed light on the problem. To explain the reason, we introduce the following matrices as in our previous paper [7]:

$$P_A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad Q_A = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \quad \text{and} \quad P_B = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

$$Q_B = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix},$$

where we assume that  $U = P_j + Q_j$  ( $j = A, B$ ) is a  $2 \times 2$  unitary matrix. Here we consider two types of the discrete-time case; one is *A-type*, the other is *B-type*. The precise definition is given in Ref. [7]. Then the unitary matrix of the discrete-time quantum walk on the line is described as

$$U_{(d)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & O & P_j & O & O & O & \dots \\ \dots & Q_j & O & P_j & O & O & \dots \\ \dots & O & Q_j & O & P_j & O & \dots \\ \dots & O & O & Q_j & O & P_j & \dots \\ \dots & O & O & O & Q_j & O & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with} \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for  $j = A$  and  $B$ . The unitary matrix  $U_{(d)}$  for the discrete-time case corresponds to  $U(t)$  for our continuous-time case at time  $t = 1$ . More generally,  $U_{(d)}^n$  corresponds to  $U(n)$  for  $n = 0, 1, \dots$ . Once an explicit formula of  $U(t)$  is obtained, the difference between continuous and discrete walks becomes clear. As we stated before,  $U(t)$  has an infinite-range form. On the other hand,  $U_{(d)}$  has a finite-range form. Moreover, we see that  $U_{(d)}$  is not homogeneous. It is believed that the difference seems to be derived from the fact that the discrete quantum walk has a coin but continuous quantum walk does not [1]. However, the situation is not so simple, since the discrete-time case also does not necessarily need the coin (see Refs. [7,10] for a more detailed discussion).

We define a continuous-time quantum walk on  $\mathbf{Z}$  by  $X_t$  whose probability distribution is defined by  $P(X_t = k) = P(k, t)$  for any location  $k \in \mathbf{Z}$  and time  $t \geq 0$ . Note that it follows from Corollary 2 that  $P(k, t) = J_k^2(t)$ . Then we obtain a new weak limit theorem for a continuous-time quantum walk on the line:

*Theorem 1.* If  $-1 \leq a < b \leq 1$ , then

$$P(a \leq X_t/t \leq b) \rightarrow \int_a^b \frac{1}{\pi\sqrt{1-x^2}} dx \quad (t \rightarrow \infty).$$

Note that

$$\int_{-1}^1 \frac{x^{2m}}{\pi\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^{\pi/2} \sin^{2m} \varphi \, d\varphi = \frac{(2m-1)!!}{(2m)!!}, \quad (1)$$

for  $m = 1, 2, \dots$ , where  $n!! = n(n-2) \cdots 5 \times 3 \times 1$ , if  $n = \text{odd}$ ,  $= n(n-2) \cdots 6 \times 4 \times 2$ , if  $n = \text{even}$ . From Theorem 1 and Eq. (1), we have

*Corollary 3.* For  $m = 1, 2, \dots$ ,

$$E((X_t/t)^{2m}) \rightarrow (2m-1)!! / (2m)!! \quad (t \rightarrow \infty).$$

By this corollary, for the standard deviation of our walk,  $\sigma_{(c)}^q(t)$ , we see that

$$\sigma_{(c)}^q(t)/t \rightarrow 1/\sqrt{2} = 0.70710\dots \quad (t \rightarrow \infty).$$

We consider a *discrete-time* quantum walk  $X_n^{(d)}$  with a symmetric distribution on the line, whose evolution is described by the Hadamard transformation (see Ref. [25]), that is,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The walk is often called the Hadamard walk. In contrast with the continuous-time case, for the Hadamard walk, the following weak limit theorem holds [6,8,9]:

*Theorem 2.* If  $-1/\sqrt{2} \leq a < b \leq 1/\sqrt{2}$ , then

$$P(a \leq X_n^{(d)}/n \leq b) \rightarrow \int_a^b \frac{1}{\pi(1-x^2)\sqrt{1-2x^2}} dx \quad (n \rightarrow \infty).$$

As a corollary, we have

$$\sigma_{(d)}^q(n)/n \rightarrow \sqrt{(2-\sqrt{2})}/2 = 0.54119\dots \quad (n \rightarrow \infty).$$

Comparing with the discrete-time case, the scaling in our continuous-time case is the same, but the limit density function is slightly different. However, both density functions have some similar properties, for example, they have two peaks at the two end points of the support.

### III. PROOF OF PROPOSITION 1

To begin with,  $A$  is rewritten as

$$A = 0 \begin{pmatrix} \dots & -1 & 0 & +1 & +2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & T & P & O & O & \dots \\ \dots & Q & T & P & O & \dots \\ \dots & O & Q & T & P & \dots \\ \dots & O & O & Q & T & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2)$$

where

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This expression corresponds to a unitary matrix for a discrete-time quantum walk in which the particle maintains its position during each time step (see [7] for more details). The following algebraic relations are useful for some computations as in the case of a discrete-time walk:

$$P^2 = Q^2 = O, \quad PT + TP = QT + TQ = I, \quad PQ + QP = T, \quad (3)$$

where  $O$  is a  $2 \times 2$  zero matrix and  $I$  is a  $2 \times 2$  unit matrix. From now on, for simplicity, Eq. (2) is written as

$$A = [\dots, O, O, O, O, Q, T, P, O, O, O, O, \dots].$$

A direct computation gives

$$A^2 = [\dots, O, O, O, O, I, 2I, I, O, O, O, O, \dots],$$

$$A^4 = [\dots, O, O, O, I, 4I, 6I, 4I, I, O, O, O, \dots],$$

$$d_0 = 0,$$

$$A^6 = [\dots, O, O, I, 6I, 15I, 20I, 15I, 6I, I, O, O, \dots].$$

It follows by induction that

$$A^{2n} = [\dots, O, O, A_{-n}^{(2n)}, \dots, A_{-1}^{(2n)}, A_0^{(2n)}, A_1^{(2n)}, \dots, A_n^{(2n)}, O, O, \dots],$$

where  $A_k^{(2n)} = A_{-k}^{(2n)} = a_k^{(2n)}I$  and

$$a_k^{(2n)} = \binom{2n}{n-k},$$

for any  $k=0, 1, \dots, n$ . On the other hand, by using  $A^{2n+1} = A^{2n} \times A$  and Eq. (3), we have

$$A^{2n+1} = [\dots, O, A_{-(n+1)}^{(2n+1)}, \dots, A_{-1}^{(2n+1)}, A_0^{(2n+1)}, A_1^{(2n+1)}, \dots, A_{n+1}^{(2n+1)}, O, \dots],$$

where

$$A_{-(n+1)}^{(2n+1)} = a_n^{(2n)}Q, \quad A_{-n}^{(2n+1)} = a_n^{(2n)}T + a_{(n-1)}^{(2n)}Q,$$

$$A_{-(n-1)}^{(2n+1)} = (a_{n-1}^{(2n)} + a_n^{(2n)})T + (a_{n-2}^{(2n)} - a_n^{(2n)})Q, \dots,$$

$$A_{-1}^{(2n+1)} = (a_1^{(2n)} + a_2^{(2n)})T + (a_0^{(2n)} - a_2^{(2n)})Q,$$

$$A_0^{(2n+1)} = (a_0^{(2n)} + a_1^{(2n)})T,$$

$$A_1^{(2n+1)} = (a_1^{(2n)} + a_2^{(2n)})T + (a_0^{(2n)} - a_2^{(2n)})P, \dots,$$

$$A_n^{(2n+1)} = a_n^{(2n)}T + a_{n-1}^{(2n)}P, \quad A_{(n+1)}^{(2n+1)} = a_n^{(2n)}P.$$

The definition of  $U(t)$  gives

$$U(t) = e^{itA/2} = \sum_{n=0}^{\infty} \frac{(it/2)^n}{n!} A^n = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n}}{(2n)!} A^{2n} + i \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+1}}{(2n+1)!} A^{2n+1}.$$

Therefore we obtain  $U(t) = B(t) + iC(t)$ , where

$$B(t) = [\dots, B_{-k}, \dots, B_{-1}, B_0, B_1, \dots, B_k, \dots],$$

$$C(t) = [\dots, C_{-k}, \dots, C_{-1}, C_0, C_1, \dots, C_k, \dots],$$

with

$$B_k = B_{-k} = b_k I, \quad b_k = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n}}{(2n)!} \binom{2n}{n-k}, \quad (k = 0, 1, \dots),$$

and

$$C_{-k} = c_k I + d_k Q, \quad C_k = c_k I + d_k P, \quad (k = 0, 1, \dots),$$

$$c_k = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+1}}{(2n+1)!} \left\{ \binom{2n}{n-k} + \binom{2n}{n-(k+1)} \right\},$$

$$(k = 0, 1, \dots),$$

$$d_k = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{2n+1}}{(2n+1)!} \left\{ \binom{2n}{n-(k-1)} + \binom{2n}{n-(k+1)} \right\}, \quad (k = 1, 2, \dots).$$

Remark that

$$J_k(x) = \left(\frac{x}{2}\right)^k \sum_{m=0}^{\infty} \frac{(ix/2)^{2m}}{(k+2m)!} \binom{k+2m}{m}, \quad (4)$$

[see Eq. (4.9.5) in Ref. [24]]. Finally, by using Eq. (4) and the following relation:

$$\binom{2n}{k} + \binom{2n}{k-1} = \binom{2n+1}{k},$$

we have the desired conclusion.

#### IV. PROOF OF THEOREM 1

We begin by stating the following result (see p. 214 in Ref. [24]): suppose that  $a$ ,  $b$ , and  $c$  are lengths of sides of a triangle and  $c^2 = a^2 + b^2 - 2ab \cos \xi$ . Then

$$J_0(c) = \sum_{k=-\infty}^{\infty} J_k(a)J_k(b)e^{ik\xi}. \quad (5)$$

If we set  $t = a = b$  in Eq. (5), then

$$J_0(t\sqrt{2(1-\cos \xi)}) = \sum_{k=-\infty}^{\infty} J_k^2(t)e^{ik\xi}. \quad (6)$$

From Eq. (6), we see that the characteristic function of a continuous-time quantum walk on the line is given by

$$E(e^{i\xi X_t}) = \sum_{k=-\infty}^{\infty} e^{ik\xi} J_k^2(t) = J_0(t\sqrt{2(1-\cos \xi)}). \quad (7)$$

First we consider that  $t$  is a positive integer case, that is,  $n (=t) = 1, 2, \dots$ . By using Eq. (7), we have

$$E(e^{i\xi X_n}) = J_0(n\sqrt{2[1-\cos(\xi/n)]}) \rightarrow J_0(\xi),$$

as  $n \rightarrow \infty$ . To know a limit density function, we use the following expression of  $J_0(x)$  (see Eq. (4.9.11) in Ref. [24]);

$$J_0(\xi) = \frac{1}{\pi} \int_0^\pi \cos(\xi \sin \varphi) d\varphi = \frac{2}{\pi} \int_0^{\pi/2} \cos(\xi \sin \varphi) d\varphi. \quad (8)$$

Taking  $x = \sin \varphi$ , we have

$$J_0(\xi) = \int_{-1}^1 \cos(\xi x) \frac{1}{\pi\sqrt{1-x^2}} dx = \int_{-1}^1 e^{i\xi x} \frac{1}{\pi\sqrt{1-x^2}} dx. \quad (9)$$

By using Eq. (8), we get

$$|J_0(\xi) - J_0(0)| \leq \frac{2}{\pi} \int_0^{\pi/2} |\cos(\xi \sin \varphi) - 1| d\varphi.$$

From the bounded convergence theorem, we see that the limit  $J_0(\xi)$  is continuous at  $\xi=0$ , since  $\cos(\xi \sin \varphi) \rightarrow 1$  as  $\xi \rightarrow 0$ . Therefore, by the continuity theorem (see p. 99 in [20]) and Eq. (9), we conclude that if  $n \rightarrow \infty$ , then  $X_n/n$  converges weakly to a random variable whose density function is given by  $1/\pi\sqrt{1-x^2}$  for  $x \in (-1, 1)$ . That is, if  $-1 \leq a < b \leq 1$ , then

$$P(a \leq X_n/n \leq b) \rightarrow \int_a^b \frac{1}{\pi\sqrt{1-x^2}} dx,$$

as  $n \rightarrow \infty$ . Next, to deal with values of  $t$  that are not integers, we want to show

$$|P(a \leq X_t/t \leq b) - P(a \leq X_{[t]}/[t] \leq b)| \rightarrow 0 \quad (t \rightarrow \infty),$$

where  $[x]$  denotes the integer part of  $x$ . To do this, we observe that

$$\begin{aligned} &|P(a \leq X_t/t \leq b) - P(a \leq X_{[t]}/[t] \leq b)| = |P(at \leq X_t \leq bt) \\ &\quad - P(a[t] \leq X_{[t]} \leq b[t])| \\ &= \left| \sum_{at \leq k \leq bt} P(X_t = k) - \sum_{a[t] \leq k \leq b[t]} P(X_{[t]} = k) \right| \\ &= \left| \sum_{at \leq k \leq bt} J_k^2(t) - \sum_{a[t] \leq k \leq b[t]} J_k^2([t]) \right| \\ &\leq \left| \left( \sum_{at \leq k \leq bt} - \sum_{a[t] \leq k \leq b[t]} \right) J_k^2(t) \right| \\ &\quad + \left| \sum_{a[t] \leq k \leq b[t]} [J_k^2(t) - J_k^2([t])] \right| = K_1(t) + K_2(t). \end{aligned}$$

To estimate  $K_1(t)$  and  $K_2(t)$ , we use the following Meissel's second expansion [see Eq. (5) on p. 228 of [23]]:

$$J_\nu(\nu \sec \beta) = \sqrt{\frac{2 \cot \beta}{\pi \nu}} e^{-P(\nu, \beta)} \cos[Q(\nu, \beta) - \pi/4], \tag{10}$$

where  $P(\nu, \beta)$  and  $Q(\nu, \beta)$  are defined by Eqs. (1) and (2), respectively, on p. 228 of [23]. First we consider the  $K_1(t)$  case. We see that

$$\begin{aligned} K_1(t) &\leq \max\{J_{[at]}^2(t), J_{[bt]}^2(t), |J_{[bt]}^2(t) - J_{[at]}^2(t)|\} \\ &\leq 2\max\{J_{[at]}^2(t), J_{[bt]}^2(t)\}. \end{aligned} \tag{11}$$

Without a loss of generality, we consider only  $J_{[ct]}^2(t)$  with  $0 < c < 1$ . From Eq. (10), taking  $\nu=[ct]$ , we have

$$\begin{aligned} J_{[ct]}^2(t) &= J_{[ct]}^2([ct] \sec \beta) = \frac{2 \cot \beta}{\pi [ct]} e^{-2P([ct], \beta)} \\ &\quad \times \cos^2[Q([ct], \beta) - \pi/4], \end{aligned} \tag{12}$$

where  $\sec \beta = t/[ct] (> 1)$ . Then, for large  $t$ , we get

$$J_{[ct]}^2(t) \leq \frac{2}{\pi [ct]} \frac{2c}{\sqrt{1-c^2}},$$

since  $\cot \beta \rightarrow c/\sqrt{1-c^2}$  as  $t \rightarrow \infty$ . The last inequality implies

$$J_{[ct]}^2(t) \rightarrow 0 \quad (t \rightarrow \infty). \tag{13}$$

Combining Eq. (11) with Eq. (13) implies that  $K_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Next we consider the  $K_2(t)$  case. Noting Eq. (13), it is enough to show that

$$\left| \sum_{[at] \leq k \leq [bt]} [J_k^2(t) - J_k^2([t])] \right| \rightarrow 0 \quad (t \rightarrow \infty).$$

As in the case of  $K_1(t)$ , fix  $0 < c < 1$ . Define  $\beta_1$  and  $\beta_2$  by  $\sec \beta_1 = t/[ct]$  and  $\sec \beta_2 = [t]/[ct]$ , respectively. Then we have

$$\begin{aligned} \cot \beta_1 &\sim \cot \beta_2 \sim \frac{c}{\sqrt{1-c^2}}, \\ \cot \beta_1 - \cot \beta_2 &\sim -\frac{c}{(1-c^2)^{3/2}} \frac{t-[t]}{t}, \\ \tan \beta_1 &\sim \tan \beta_2 \sim \frac{\sqrt{1-c^2}}{c}, \\ \tan \beta_1 - \tan \beta_2 &\sim \frac{1}{c\sqrt{1-c^2}} \frac{t-[t]}{t}, \\ \beta_1 &\sim \beta_2 \sim \arccos c, \\ \beta_1 - \beta_2 &\sim \frac{(2-\sqrt{1-c^2})c}{2\sqrt{1-c^2}} \frac{t-[t]}{t}, \end{aligned}$$

where  $f(t) \sim g(t)$  means  $f(t)/g(t) \rightarrow 1$  as  $t \rightarrow \infty$ . By using the above estimates, Eq. (12), and noting  $P([ct], \beta_i) \rightarrow 0 (t \rightarrow \infty)$ ,  $Q([ct], \beta_i) \sim [ct](\tan \beta_i - \beta_i)$  for  $i=1, 2$ , we see that

$$\begin{aligned} &J_{[ct]}^2(t) - J_{[ct]}^2([t]) = J_{[ct]}^2([ct] \sec \beta_1) - J_{[ct]}^2([ct] \sec \beta_2) \\ &\sim \frac{2}{\pi [ct]} (\cot \beta_1 \cos^2\{[ct](\tan \beta_1 - \beta_1) - \pi/4\} \\ &\quad - \cot \beta_2 \cos^2\{[ct](\tan \beta_2 - \beta_2) - \pi/4\}) \\ &\sim \frac{1}{\pi [ct]} (\cot \beta_1 - \cot \beta_2 + \cot \beta_1 \\ &\quad \times \sin\{2[ct](\tan \beta_1 - \beta_1)\} - \cot \beta_2 \\ &\quad \times \sin\{2[ct](\tan \beta_2 - \beta_2)\}) \sim \frac{1}{\pi [ct]} \\ &\quad \times [(\cot \beta_1 - \cot \beta_2)(1 + \sin\{2[ct](\tan \beta_1 - \beta_1)\}) \\ &\quad + \cot \beta_2(\sin\{2[ct](\tan \beta_1 - \beta_1)\} \\ &\quad - \sin\{2[ct](\tan \beta_2 - \beta_2)\})] \end{aligned}$$

$$\begin{aligned} &\sim \frac{1}{\pi[ct]} [(\cot \beta_1 - \cot \beta_2)(1 + \sin\{2[ct](\tan \beta_1 - \beta_1)\}) \\ &- \cot \beta_2 \times 2[ct]\{(\tan \beta_1 - \tan \beta_2) - (\beta_1 - \beta_2)\}\cos\{2[ct] \\ &\times (\tan \beta_1 - \beta_1)\}]. \end{aligned}$$

$$\begin{aligned} J_{[ct]}^2(t) - J_{[ct]}^2([t]) &\sim -\frac{t-[t]}{\pi(1-c^2)t^2} \\ &\times (C_1(c)\{1 + \sin[\phi(c,t)]\} \\ &+ t C_2(c)\cos[\phi(c,t)]), \end{aligned} \quad (14)$$

where  $C_1(c) = 1/\sqrt{1-c^2}$ ,  $C_2(c) = 2 - (2 - \sqrt{1-c^2})c^2$ , and  $\phi(c,t) = 2[ct](\sqrt{1-c^2}/c - \arccos c)$ . For simplicity, we suppose that  $0 < a < b \leq 1$ . Noting Eq. (13), we see that

Therefore we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} K_2(t) &= \lim_{t \rightarrow \infty} \left| \sum_{k=0}^{[bt]-[at]} \{J_{[at]+k}^2(t) - J_{[at]+k}^2([t])\} \right| \\ &= \lim_{t \rightarrow \infty} (b-a)t \left| \int_0^1 \{J_{[a+(b-a)x]t}^2(t) - J_{[a+(b-a)x]t}^2([t])\} dx \right| \\ &\leq \lim_{t \rightarrow \infty} \frac{b-a}{\pi t} \int_0^1 \left| \frac{C_1(a+(b-a)x)(1 + \sin\{\phi(a+(b-a)x,t)\})}{1 - [a+(b-a)x]^2} \right| dx \\ &\quad + \lim_{t \rightarrow \infty} \frac{b-a}{\pi} \left| \int_0^1 \frac{C_2(a+(b-a)x)}{1 - [a+(b-a)x]^2} \cos[\phi(a+(b-a)x,t)] dx \right| \\ &= \lim_{t \rightarrow \infty} K_3(t) + \lim_{t \rightarrow \infty} K_4(t). \end{aligned}$$

To get the third inequality, we used Eq. (14). It is easily obtained that  $\lim_{t \rightarrow \infty} K_3(t) = 0$  since the integral is bounded above by a constant which is independent of  $t$ . On the other hand, it follows from the Riemann-Lebesgue lemma that  $\lim_{t \rightarrow \infty} K_4(t) = 0$ . So we have  $\lim_{t \rightarrow \infty} K_2(t) = 0$ . Thus the proof of Theorem 1 is complete.

### V. CONCLUSION AND DISCUSSION

In contrast with the classical random walk for which the central limit theorem holds, we have shown a weak limit theorem  $X_t^{(c)}/t \rightarrow dx/\pi\sqrt{1-x^2}$  ( $t \rightarrow \infty$ ) for the continuous-time quantum walk on the line (Theorem 1). Interestingly, although the definition of the walk is very different from that of the discrete-time one, the limit theorem resembles that of the discrete-time case, for example, the symmetric Hadamard walk:  $X_n^{(d)}/n \rightarrow dx/\pi(1-x^2)\sqrt{1-2x^2}$  ( $n \rightarrow \infty$ ).

Very recently, Romanelli *et al.* [12] investigated a continuum time limit for a discrete-time quantum walk on  $\mathbf{Z}$  and obtained the position probability distribution. When the initial condition is given by  $\tilde{a}_l(0) = \delta_{l,0}$ ,  $\tilde{b}_l(0) \equiv 0$  in their notation for the Hadamard walk, the distribution at lo-

cation  $k$  and time  $t$  becomes the following in our notation:  $P_{(R)}(k,t) = J_k^2(t/\sqrt{2})$ . More generally, we consider the time evolution given by the following unitary matrix:

$$U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},$$

where  $\theta \in (0, \pi/2)$ . Note that the  $\theta = \pi/4$  case is equivalent to the Hadamard walk. Then we have  $P_{(R,\theta)}(k,t) = J_k^2(t \cos \theta)$ . In this case, a similar argument in the proof of Theorem 1 implies that if  $-\cos \theta \leq a < b \leq \cos \theta$ , then

$$P(a \leq X_t^{(R,\theta)}/t \leq b) \rightarrow \int_a^b \frac{1}{\pi\sqrt{\cos^2 \theta - x^2}} dx \quad (t \rightarrow \infty),$$

where  $X_t^{(R,\theta)}$  denotes a continuous-time quantum walk whose probability distribution is given by  $P_{(R,\theta)}(k,t)$ . As a consequence, we obtain

$$E((X_t^{(R,\theta)}/t)^{2m}) \rightarrow \cos^{2m} \theta (2m-1)!!/(2m)!! \quad (t \rightarrow \infty).$$

In particular, when  $m=1$ , the limit  $\cos^2 \theta/2$  is consistent with Eq. (30) in Romanelli *et al.* [12]



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